Coefficients of the poles of local zeta functions and their applications to oscillating integrals *

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Abstract

We introduce a new method which enables us to calculate the coefficients of the poles of local zeta functions very precisely and prove some explicit formulas. Some vanishing theorems for the candidate poles of local zeta functions will be also given. Moreover we apply our method to oscillating integrals and obtain an explicit formula for the coefficients of their asymptotic expansions.

1 Introduction

The theory of local zeta functions is an important subject in many fields of mathematics, such as generalized functions, number theory and prehomogeneous vector spaces. By the fundamental work [18] due to Varchenko, we can find a subset P of $\mathbb{Q}_{<0}$ in which the poles of a local zeta function are contained (see Section 3 and [1] etc. for the detail). After this pioneering paper [18] many authors studied the poles of local zeta functions. However, to the best of our knowledge, there is almost no paper which calculated the coefficients of these poles explicitly in a general setting. Note that in a remarkable work [7] Igusa could calculate these coefficients when the local zeta functions are associated to the relative invariants of some prehomogeneous vector spaces. He calculated the coefficients by using some group actions.

In this paper, we propose a new method which enables us to calculate the coefficients of the poles of local zeta functions as precisely as we want. The key idea is the use of a meromorphic continuation of the distribution

$$x_{1+}^{l_1\lambda+m_1} x_{2+}^{l_2\lambda+m_2} \cdots x_{n+}^{l_n\lambda+m_n} \in \mathcal{D}'(\mathbb{R}^n) \tag{1.1}$$

($l_1, l_2, \dots, l_n \in \mathbb{R}_{>0}$ and $m_1, m_2, \dots, m_n \in \mathbb{R}_+ = \mathbb{R}_{\geq 0}$) with respect to the complex parameter λ, which is different from the usual one used in the study of local zeta functions (see [1] etc.). See Section 2 for the detail. This meromorphic continuation is useful and enables us to get very precise information on the poles of local zeta functions. In order to state our results, now let us recall the definition of local zeta functions. Let f be a real-valued

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real analytic function defined on an open neighborhood U of $0 \in \mathbb{R}^n$ and $\varphi \in C_0^\infty(U)$ a real-valued test function $\varphi \in C_0^\infty(U)$ on U. Then the integral

$$Z_f(\varphi)(\lambda) = \int_{\mathbb{R}^n} |f(x)|^\lambda \varphi(x) dx \tag{1.2}$$

converges locally uniformly on $\{\lambda \in \mathbb{C} \mid \Re\lambda > 0\}$ and defines a holomorphic function there. Moreover $Z_f(\varphi)(\lambda)$ can be extended to a meromorphic function on the whole complex plane $\mathbb{C}$ (see [1] and [18] etc.). This meromorphic function $Z_f(\varphi)$ of λ is called the local zeta function associated to f and φ. As is clear from the proofs of our results in this paper, by our method we can calculate the coefficients of the poles of $Z_f(\varphi)$ very precisely once a good resolution of the singularities of the hypersurface $\{x \in U \mid f(x) = 0\}$ is obtained. It would be possible to calculate them as precisely as we want by numerical computations. However, since the general formula (which is evident from our proofs) is involved, we restrict ourselves here to a generic case where the formula can be stated neatly. From now on, assume that the hypersurface $\{x \in U \mid f(x) = 0\}$ has an isolated singular point at $0 \in U \subset \mathbb{R}^n$ and f is convenient and non-degenerate (in the sense of Definition 3.2). Let Σ_0 be the dual fan of the Newton polygon $\Gamma_+(f)$ of f and Σ a smooth subdivision of Σ_0. Recall that $\Sigma = \{\sigma\}$ is a family of rational polyhedral convex cones in $\mathbb{R}^n_+ = \mathbb{R}^n_{\geq 0}$ such that $\mathbb{R}^n_+ = \bigcup_{\sigma \in \Sigma} \sigma$. For a cone σ in Σ let $\{a^1(\sigma), a^2(\sigma), \dots, a^{\mathrm{dim}\sigma}(\sigma)\} \subset \partial\sigma \cap (\mathbb{Z}^n \setminus \{0\})$ be the 1-skelton of σ. For each $1 \leq i \leq \mathrm{dim}\sigma$ set

$$l(a^i(\sigma)) = \min_{\alpha \in \Gamma_+(f)} \langle a^i(\sigma), \alpha \rangle \in \mathbb{Z}_+ \tag{1.3}$$

and

$$|a^i(\sigma)| = \sum_{j=1}^n a^i(\sigma)_j \in \mathbb{Z}_{>0}. \tag{1.4}$$

For $0 \leq k \leq n$ let $\Sigma^{(k)} \subset \Sigma$ be the subset of Σ consisting of k-dimensional cones.

Definition 1.1. *(see [1] and [18] etc.) Let $P \subset \mathbb{Q}_{<0}$ be the union of the following subsets of $\mathbb{Q}_{<0}$:*

$$\{-1, -2, -3, \cdots\cdots\}, \tag{1.5}$$

$$\left\{ -\frac{|a^1(\sigma)|}{l(a^1(\sigma))}, -\frac{|a^1(\sigma)|+1}{l(a^1(\sigma))}, \cdots\cdots \right\} \qquad (\sigma \in \Sigma^{(1)} \text{ such that } l(a^1(\sigma)) > 0). \tag{1.6}$$

By the fundamental results of [18], the poles of $Z_f(\varphi)$ are contained in P. We call an element of P a candidate pole of $Z_f(\varphi)$. Let us order the candidate poles of $Z_f(\varphi)$ as

$$P = \{-\lambda_1 > -\lambda_2 > -\lambda_3 > \cdots\cdots\} \qquad (\lambda_j \in \mathbb{Q}_{>0}). \tag{1.7}$$

Definition 1.2. *(i) Let $\sigma \in \Sigma$. For $1 \leq i \leq \mathrm{dim}\sigma$ such that $l(a^i(\sigma)) > 0$ ($\Longleftrightarrow a^i(\sigma) \in \mathbb{R}^n_{>0}$) we set*

$$K^i(\sigma) = \left\{ \frac{|a^i(\sigma)|}{l(a^i(\sigma))}, \frac{|a^i(\sigma)|+1}{l(a^i(\sigma))}, \cdots\cdots \right\} \subset \mathbb{Q}_{>0}. \tag{1.8}$$

(ii) For a candidate pole $-\lambda_j \in P$ of $Z_f(\varphi)$ and $0 \leq k \leq n$ we define a subset $\Sigma_j^{(k)}$ of $\Sigma^{(k)}$ by

$$\Sigma_j^{(k)} = \{\sigma \in \Sigma^{(k)} \mid l(a^i(\sigma)) > 0 \quad \text{and} \quad \lambda_j \in K^i(\sigma) \quad \text{for} \quad 1 \leq i \leq k\}. \tag{1.9}$$

(iii) For $\sigma \in \Sigma_j^{(k)}$ and $1 \leq i \leq k$ we define a non-negative integer $\nu(\sigma)_i \in \mathbb{Z}_+$ by

$$\lambda_j = \frac{|a^i(\sigma)| + \nu(\sigma)_i}{l(a^i(\sigma))}. \tag{1.10}$$

For the sake of simplicity, in this introduction we assume that $-\lambda_j \in P$ is not an integer. Then it is well-known that the order of the pole of $Z(\varphi)$ at $\lambda = -\lambda_j$ is less than or equal to

$$k_j := \max\{0 \leq k \leq n \mid \Sigma_j^{(k)} \neq \emptyset\} \in \mathbb{Z}_+. \tag{1.11}$$

Let

$$\frac{a_{j,k_j}(\varphi)}{(\lambda+\lambda_j)^{k_j}} + \cdots\cdots + \frac{a_{j,2}(\varphi)}{(\lambda+\lambda_j)^2} + \frac{a_{j,1}(\varphi)}{(\lambda+\lambda_j)} + \cdots\cdots \qquad (a_{j,k}(\varphi) \in \mathbb{R}) \tag{1.12}$$

be the Laurent expansion of $Z_f(\varphi)$ at $\lambda = -\lambda_j$. Then we obtain the following vanishing theorem which generalizes that of Jacobs in [8].

Theorem 1.3. *Let $1 \leq k \leq k_j$. Assume that for any $\sigma \in \Sigma_j^{(k)}$ there exists $1 \leq i \leq k$ such that $\nu(\sigma)_i$ is odd. Then we have $a_{j,k}(\varphi) = \cdots = a_{j,k_j}(\varphi) = 0$.*

We can state also another vanishing theorem. In order to state it, let

$$\varphi(x) = \sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha x^\alpha \qquad (c_\alpha \in \mathbb{R}) \tag{1.13}$$

be the Taylor expansion of the test function φ at $0 \in U \subset \mathbb{R}^n$.

Theorem 1.4. *For $1 \leq k \leq k_j$ assume that $\{\alpha \in \mathbb{Z}_+^n \mid c_\alpha \neq 0\} \cap \Delta_{j,k} = \emptyset$, where $\Delta_{j,k}$ is a certain compact subset of $\mathbb{R}_+^n$ (see Definition 3.7). Then we have $a_{j,k}(\varphi) = \cdots = a_{j,k_j}(\varphi) = 0$.*

Moreover, for the coefficients $a_{j,n}(\varphi)$ of the deepest poles $\lambda = -\lambda_j \in P$ we can prove the following explicit formula. For $\sigma \in \Sigma_j^{(n)}$ and $\alpha \in \mathbb{Z}_+^n$ we define an integer $\mu(\sigma,\alpha)_i$ by

$$\mu(\sigma,\alpha)_i = \nu(\sigma)_i - \langle a^i(\sigma), \alpha \rangle \in \mathbb{Z}. \tag{1.14}$$

Theorem 1.5. *Assume that $k_j = n$. Then $a_{j,n}(\varphi)$ is given by*

$$\begin{aligned} &a_{j,n}(\varphi) \\ = \quad &\sum_{\alpha \in \Delta_{j,n}} \left\{ \sum_{\sigma \in \Sigma_j^{(n)}} \left(\prod_{i=1}^n \frac{1+(-1)^{\nu(\sigma)_i}}{l(a^i(\sigma)) \times \mu(\sigma,\alpha)_i!} \right) \frac{\partial^{\mu(\sigma,\alpha)_1+\cdots+\mu(\sigma,\alpha)_n}}{\partial y_1^{\mu(\sigma,\alpha)_1} \cdots \partial y_n^{\mu(\sigma,\alpha)_n}} |f_\sigma|^{-\lambda_j}(0) \right\} \\ \times \quad &\frac{\partial_x^\alpha \varphi(0)}{\alpha!}, \end{aligned} \tag{1.15}$$

where f_σ is a function defined in a neighborhood of $0 \in \mathbb{R}_y^n$ such that $f_\sigma(0) \neq 0$ (see Section 3 for the definition) and for $\mu < 0$ we set $\frac{\partial^\mu}{\partial y_i^\mu}(\cdot) = 0$.

Our results on the poles of local zeta function $Z_f(\varphi)$ also have some applications to oscillating integral $I_f(\varphi)(t)$ ($t \in \mathbb{R}$) defined by

$$I_f(\varphi)(t) = \int_{\mathbb{R}^n} e^{itf(x)} \varphi(x) dx. \tag{1.16}$$

By the fundamental results of Varchenko [18] (see also [1] for the detail), as $t \longrightarrow +\infty$ the oscillating integral $I_f(\varphi)(t)$ has an asymptotic expansion of the form

$$I_f(\varphi)(t) \sim \sum_{j=1}^{\infty} \sum_{k=1}^{k_j} c_{j,k}(\varphi) t^{-\lambda_j} (\log t)^{k-1}, \tag{1.17}$$

where $c_{j,k}(\varphi)$ are some complex numbers. Despite the important contributions by many mathematicians (see for example [1], [2], [6] and [17] etc.), little is known about the coefficients $c_{j,k}(\varphi)$ of this asymptotic expansion. Here we can prove the following vanishing theorem.

Theorem 1.6. *(i) Let $1 \leq k \leq k_j$. Assume that λ_j is not an integer and $\{\alpha \in \mathbb{Z}^n_+ \mid c_\alpha \neq 0\} \cap \Delta_{j,k} = \emptyset$. Then we have $c_{j,k}(\varphi) = \cdots = c_{j,k_j}(\varphi) = 0$.*
(ii) Let $2 \leq k \leq k_j$. Assume that λ_j is an integer and $\{\alpha \in \mathbb{Z}^n_+ \mid c_\alpha \neq 0\} \cap \Delta_{j,k-1} = \emptyset$. Then we have $c_{j,k}(\varphi) = \cdots = c_{j,k_j}(\varphi) = 0$.

Moreover by Theorem 1.5 we will prove also an explicit formula for the coefficients $c_{j,n}(\varphi)$ of $t^{-\lambda_j}(\log t)^{n-1}$ in the asymptotic expansion of $I_f(\varphi)$. See Section 5 for the detail. Finally let us mention that the method introduced in this paper would have some applications also in the study of p-adic local zeta functions (see [7] etc.). It would be a very interesting subject to study the twisted monodromy conjecture (see [13] for a review on this conjecture) by this method.

2 Meromorphic continuations of distributions

In this section, we prepare some basic results on the meromorphic continuations of the distributions $x_{1+}^{l_1\lambda+m_1} x_{2+}^{l_2\lambda+m_2} \cdots x_{n+}^{l_n\lambda+m_n}$ ($l_i \in \mathbb{R}_{>0}$, $m_i \in \mathbb{R}_+ = \mathbb{R}_{\geq 0}$) with respect to the complex parameter λ. In Section 3 and 4 these results will be used effectively to study the poles of local zeta functions. First, let us recall the classical result in the 1-dimensional case (see [4] etc.). Let $l \in \mathbb{R}_{>0}$ be a positive real number and $m \in \mathbb{R}_+$. Then for $\varphi \in C_0^\infty(\mathbb{R})$ the integral

$$F_+(\varphi)(\lambda) = \int_{-\infty}^{+\infty} x_+^{l\lambda+m} \varphi(x) dx = \int_0^{+\infty} x^{l\lambda+m} \varphi(x) dx \tag{2.1}$$

converges locally uniformly on $\{\lambda \in \mathbb{C} \mid \Re\lambda > -\frac{m+1}{l}\}$ and defines a holomorphic function there. In other words, if $\Re\lambda > -\frac{m+1}{l}$ the map $\varphi \mapsto F_+(\varphi)(\lambda) \in \mathbb{R}$ defines a distribution $x_+^{l\lambda+m} \in \mathcal{D}'(\mathbb{R})$ on $\mathbb{R}$. Let us fix a test function $\varphi \in C_0^\infty(\mathbb{R})$. Following the methods in Gelfand-Shilov [6] we shall extend $F_+(\varphi)$ to a meromorphic function on the whole complex

plane $\mathbb{C}$ as follows. First take a sufficiently large integer $N >> 0$. Then for any $\lambda \in \mathbb{C}$ such that $\Re\lambda > -\frac{m+1}{l}$ we have

$$\begin{aligned} F_+(\varphi)(\lambda) &= \int_0^{+\infty} x^{l\lambda+m}\varphi(x)dx \\ &= \int_0^1 x^{l\lambda+m}\left[\varphi(x) - \sum_{r=1}^{N}\varphi^{(r-1)}(0)\frac{x^{r-1}}{(r-1)!}\right]dx \\ &+ \int_1^{+\infty} x^{l\lambda+m}\varphi(x)dx + \sum_{r=1}^{N}\frac{\varphi^{(r-1)}(0)}{(r-1)!(l\lambda+m+r)} \\ &= \int_0^1 x^{l\lambda+m}dx\int_0^x \frac{\varphi^{(N)}(t)}{(N-1)!}(x-t)^{N-1}dt + \int_1^{+\infty} x^{l\lambda+m}\varphi(x)dx \\ &+ \sum_{r=1}^{N}\frac{\varphi^{(r-1)}(0)}{(r-1)!\times l(\lambda+\frac{m+r}{l})} \\ &= \int_0^1 g_N(\lambda,t)\varphi^{(N)}(t)dt + \int_1^{+\infty} t^{l\lambda+m}\varphi(t)dt \\ &+ \sum_{r=1}^{N}\frac{1}{(r-1)!\times l(\lambda+\frac{m+r}{l})}\langle(-1)^{r-1}\delta^{(r-1)},\varphi\rangle, \end{aligned} \tag{2.2}$$

where $\delta \in \mathcal{D}'(\mathbb{R})$ is Dirac's delta function and we set

$$g_N(\lambda,t) = \frac{1}{(N-1)!}\int_t^1 x^{l\lambda+m}(x-t)^{N-1}dx \tag{2.3}$$

for $0 < t \leq 1$. The function $g_N(\lambda,t)$ satisfies the following nice properties.

Lemma 2.1. *(i) For any $0 < t \leq 1$, $g_N(\lambda,t)$ is an entire function of λ.*
(ii) If $\Re\lambda > -\frac{m+N+1}{l}$ then the function $g_N(\lambda,t)$ of t is integrable on $(0,1]$.

By this lemma we see that the integral

$$\int_0^1 g_N(\lambda,t)\varphi^{(N)}(t)dt \tag{2.4}$$

converges locally uniformly on $\{\lambda \in \mathbb{C} \mid \Re\lambda > -\frac{m+N+1}{l}\}$ and defines a holomorphic function there. Since the integral $\int_1^{+\infty} t^{l\lambda+m}\varphi(t)dt$ is an entire function of λ, the function $F_+(\varphi)$ is extended to a meromorphic function on $\{\lambda \in \mathbb{C} \mid \Re\lambda > -\frac{m+N+1}{l}\}$ with simple poles at $\lambda = -\frac{m+r}{l}$ $(r = 1,2,\ldots,N)$. Moreover the residue of $F_+(\varphi)$ at $\lambda = -\frac{m+r}{l}$ is given by

$$\mathrm{Res}(F_+(\varphi); -\frac{m+r}{l}) = \frac{1}{(r-1)!\times l}\langle(-1)^{r-1}\delta^{(r-1)},\varphi\rangle. \tag{2.5}$$

Similarly set

$$F_-(\varphi)(\lambda) = \int_{-\infty}^{+\infty} x_-^{l\lambda+m}\varphi(x)dx = \int_{-\infty}^{0} |x|^{l\lambda+m}\varphi(x)dx. \tag{2.6}$$

Then $F_-(\varphi)(\lambda)$ can be also extended to a meromorphic function on the complex plane $\mathbb{C}$ with simple poles at $\lambda = -\frac{m+r}{l}$ $(r = 1, 2, \ldots, N)$ and we have

$$\mathrm{Res}(F_-(\varphi); -\frac{m+r}{l}) = \frac{1}{(r-1)! \times l} \langle \delta^{(r-1)}, \varphi \rangle. \tag{2.7}$$

This basic result in the 1-dimensional case can be generalized to higher-dimensional cases as follows. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be a test function on $\mathbb{R}^n$. For positive real numbers $l_1, l_2, \ldots, l_n \in \mathbb{R}_{>0}$ and $m_1, m_2, \ldots, m_n \in \mathbb{R}_+$ we set

$$G(\varphi)(\lambda) = \int_{\mathbb{R}^n} x_{1+}^{l_1\lambda+m_1} x_{2+}^{l_2\lambda+m_2} \cdots x_{n+}^{l_n\lambda+m_n} \varphi(x) dx \tag{2.8}$$

and

$$L = -\min \left\{ \frac{m_1+1}{l_1}, \frac{m_2+1}{l_2}, \cdots, \frac{m_n+1}{l_n} \right\}. \tag{2.9}$$

Then this integral converges locally uniformly on $\{\lambda \in \mathbb{C} \mid \Re\lambda > L\}$ and defines a holomorphic function there. By using the tensor product $\otimes$ of distributions we can rewrite $G(\varphi)(\lambda)$ as

$$G(\varphi)(\lambda) = \langle x_{1+}^{l_1\lambda+m_1} \otimes x_{2+}^{l_2\lambda+m_2} \otimes \cdots \otimes x_{n+}^{l_n\lambda+m_n}, \varphi \rangle. \tag{2.10}$$

Let $N >> 0$ be a sufficiently large integer. Then for $\Re\lambda > L$ we have the following equalities in the space $\mathcal{D}'(\mathbb{R})$ of 1-dimensional distributions.

$$x_{i+}^{l_i\lambda+m_i} = G_{i,N}(\lambda) + \sum_{r=1}^{N} \frac{(-1)^{r-1}}{(r-1)! l_i(\lambda + \frac{m_i+r}{l_i})} \delta^{(r-1)}(x_i) \qquad (i = 1, 2, \ldots, n), \tag{2.11}$$

where $G_{i,N}(\lambda) \in \mathcal{D}'(\mathbb{R})$ is a 1-dimensional distribution defined by

$$\langle G_{i,N}(\lambda), \phi \rangle = \int_0^1 g_{i,N}(\lambda, t) \phi^{(N)}(t) dt + \int_1^{+\infty} t^{l_i\lambda+m_i} \phi(t) dt \qquad (\phi \in C_0^\infty(\mathbb{R})). \tag{2.12}$$

Here $g_{i,N}(\lambda, t)$ is an integrable function of $t \in (0, 1]$ for $\lambda \in \mathbb{C}$ such that $\Re\lambda > \frac{m_i+N+1}{l_i}$. Putting this new expressions of the 1-dimensional distributions $x_{i+}^{l_i\lambda+m_i}$ into (2.10) we see that $G(\varphi)(\lambda)$ is extended to a meromorphic function on $\{\lambda \in \mathbb{C} \mid \Re\lambda > L_N\}$, where we set

$$L_N = -\min \left\{ \frac{m_1+N+1}{l_1}, \frac{m_2+N+1}{l_2}, \cdots, \frac{m_n+N+1}{l_n} \right\}. \tag{2.13}$$

Since the integer $N >> 0$ can be taken as large as possible, $G(\varphi)(\lambda)$ is meromorphically continued to the whole complex plane $\mathbb{C}$. Moreover the poles of this meromorphic function $G(\varphi)(\lambda)$ are contained in a discrete set P in $\mathbb{C}$ defined by

$$P = \bigcup_{1 \le i \le n} \left\{ -\frac{m_i+1}{l_i}, -\frac{m_i+2}{l_i}, -\frac{m_i+3}{l_i}, \cdots\cdots \right\} \subset \mathbb{R}_{<0} \subset \mathbb{C}. \tag{2.14}$$

An element of P is called a candidate pole of $G(\varphi)$. Let us rewrite this set P as

$$P = \{-\lambda_1 > -\lambda_2 > -\lambda_3 > \cdots\cdots\} \qquad (\lambda_j \in \mathbb{R}_{>0}). \tag{2.15}$$

For each candidate pole $-\lambda_j \in P$ of $G(\varphi)$ we define a subset S_j of $\{1, 2, \ldots, n\}$ by

$$S_j = \{1 \le i \le n \mid {}^{\exists} r \in \mathbb{Z}_{>0} \text{ such that } \frac{m_i + r}{l_i} = \lambda_j\} \tag{2.16}$$

and set $k_j = \sharp S_j$. Then we can easily see that the order of the pole of $G(\varphi)$ at $\lambda = -\lambda_j$ is less than or equal to k_j. For a candidate pole $-\lambda_j \in P$ of $G(\varphi)$ let

$$\frac{a_{j,k_j}}{(\lambda+\lambda_j)^{k_j}} + \cdots\cdots + \frac{a_{j,2}}{(\lambda+\lambda_j)^2} + \frac{a_{j,1}}{(\lambda+\lambda_j)} + \cdots\cdots \qquad (a_{j,k} \in \mathbb{R}) \tag{2.17}$$

be the Laurent expansion of $G(\varphi)$ at $\lambda = -\lambda_j$. For each $i \in S_j \subset \{1, 2, \ldots, n\}$ we define a non-negative integer $\nu_i \in \mathbb{Z}_+$ by the formula

$$\frac{m_i + 1 + \nu_i}{l_i} = \lambda_j. \tag{2.18}$$

Proposition 2.2. *Let $1 \le k \le k_j$. Then the coefficient $a_{j,k}$ of $\frac{1}{(\lambda+\lambda_j)^k}$ in the Laurent expansion of $G(\varphi)(\lambda)$ at $\lambda = -\lambda_j$ is written as*

$$a_{j,k} = \sum_{S \subset S_j : \sharp S \ge k} \frac{1}{(\sharp S - k)!} \left\{ \frac{\partial^{\sharp S - k}}{\partial \lambda^{\sharp S - k}} \rho_S(\lambda) \right\}_{\lambda = -\lambda_j}, \tag{2.19}$$

where for each subset $S \subset S_j$ of S_j such that $\sharp S \ge k$ the function $\rho_S(\lambda)$ is holomorphic at $\lambda = -\lambda_j$ and written as follows:
For the sake of simplicity, assume that $S = \{1, 2, \ldots, l\}$ for some $l \ge k$. Then we have

$$\begin{aligned} \rho_S(\lambda) &= \prod_{i=1}^{l} \frac{1}{l_i \nu_i!} \times \int_{\{x \in \mathbb{R}^n \mid x_1 = \cdots = x_l = 0\}} \prod_{i=l+1}^{n} g_i(\lambda, x_i) \\ &\quad \times \left\{ \frac{\partial^{\nu_1 + \cdots + \nu_l}}{\partial x_1^{\nu_1} \cdots \partial x_l^{\nu_l}} \varphi(x) \right\}_{x_1 = \cdots = x_l = 0} dx_{l+1} \cdots dx_n, \end{aligned} \tag{2.20}$$

where $g_i(\lambda, \cdot)$ $(i = l+1, \ldots, n)$ are 1-dimensional integrable functions with holomorphic parameter λ at $\lambda = -\lambda_j$.

When $k_j = \sharp S_j = n$ we have the following very simple expression of $a_{j,n}$.

Proposition 2.3. *If $k_j = \sharp S_j = n$, we have*

$$a_{j,n} = \left(\prod_{i=1}^{n} \frac{1}{l_i \nu_i!} \right) \frac{\partial^{\nu_1 + \cdots + \nu_n}}{\partial x_1^{\nu_1} \cdots \partial x_n^{\nu_n}} \varphi(0). \tag{2.21}$$

Similarly let us set

$$H(\varphi)(\lambda) = \int_{\mathbb{R}^n} |x_1|^{l_1 \lambda + m_1} |x_2|^{l_2 \lambda + m_2} \cdots |x_n|^{l_n \lambda + m_n} \varphi(x) dx \tag{2.22}$$

Then $H(\varphi)$ can be also extended to a meromorphic function on $\mathbb{C}$ whose poles are contained in the set $P \subset \mathbb{R}_{<0}$. Moreover the order of the pole of $H(\varphi)$ at $\lambda = -\lambda_j \in P$ is less than or equal to k_j. For a candidate pole $-\lambda_j \in P$ of $H(\varphi)$ let

$$\frac{b_{j,k_j}}{(\lambda+\lambda_j)^{k_j}} + \cdots\cdots + \frac{b_{j,2}}{(\lambda+\lambda_j)^2} + \frac{b_{j,1}}{(\lambda+\lambda_j)} + \cdots\cdots \qquad (b_{j,k} \in \mathbb{R}) \tag{2.23}$$

be the Laurent expansion of $H(\varphi)$ at $\lambda = -\lambda_j$.

Proposition 2.4. *Let $1 \leq k \leq k_j$. Then the coefficient $b_{j,k}$ of $\frac{1}{(\lambda+\lambda_j)^k}$ in the Laurent expansion of $H(\varphi)(\lambda)$ at $\lambda = -\lambda_j$ is written as*

$$b_{j,k} = \sum_{S \subset S_j : \sharp S \geq k} \frac{1}{(\sharp S - k)!} \left\{ \frac{\partial^{\sharp S - k}}{\partial \lambda^{\sharp S - k}} \tau_S(\lambda) \right\}_{\lambda = -\lambda_j}, \tag{2.24}$$

where for each subset $S \subset S_j$ of S_j such that $\sharp S \geq k$ the function $\tau_S(\lambda)$ is holomorphic at $\lambda = -\lambda_j$ and written as follows:
For the sake of simplicity, assume that $S = \{1, 2, \ldots, l\}$ for some $l \geq k$. Then we have

$$\begin{aligned} \tau_S(\lambda) &= \prod_{i=1}^{l} \frac{1 + (-1)^{\nu_i}}{l_i \nu_i!} \times \int_{\{x \in \mathbb{R}^n | x_1 = \cdots = x_l = 0\}} \prod_{i=l+1}^{n} g_i(\lambda, x_i) \\ &\quad \times \left\{ \frac{\partial^{\nu_1 + \cdots + \nu_l}}{\partial x_1^{\nu_1} \cdots \partial x_l^{\nu_l}} \varphi(x) \right\}_{x_1 = \cdots = x_l = 0} dx_{l+1} \cdots dx_n, \end{aligned} \tag{2.25}$$

where $g_i(\lambda, \cdot)$ $(i = l+1, \ldots, n)$ are as in Proposition 2.2.

As a special case of this proposition, we obtain the following.

Corollary 2.5. *Let $1 \leq k \leq k_j$. Assume that for any $S \subset S_j$ with $\sharp S = k$ there exists $i \in S$ such that ν_i is odd. Then we have $b_{j,k} = \cdots = b_{j,k_j} = 0$.*

If $k_j = \sharp S_j = n$, we can also obtain the following explicit expression of $b_{j,n}$.

Proposition 2.6. *If $k_j = \sharp S_j = n$, we have*

$$b_{j,n} = \left(\prod_{i=1}^{n} \frac{1 + (-1)^{\nu_i}}{l_i \nu_i!} \right) \frac{\partial^{\nu_1 + \cdots + \nu_n}}{\partial x_1^{\nu_1} \cdots \partial x_n^{\nu_n}} \varphi(0). \tag{2.26}$$

3 Vanishing theorems for the poles of local zeta functions

Let f be a real-valued real analytic function defined on an open neighborhood U of $0 \in \mathbb{R}^n$. Then for any real-valued test function $\varphi \in C_0^\infty(U)$ the integral

$$Z_f(\varphi)(\lambda) = \int_{\mathbb{R}^n} |f(x)|^\lambda \varphi(x) dx \tag{3.1}$$

converges locally uniformly on $\{\lambda \in \mathbb{C} \mid \Re\lambda > 0\}$ and defines a holomorphic function there. Moreover it is well-known that $Z_f(\varphi)(\lambda)$ can be extended to a meromorphic function defined on the whole complex plane $\mathbb{C}$ (see [1], [9] and [18] etc.). In this section, assuming that the real hypersurface $\{x \in U \mid f(x) = 0\}$ has an isolated singular point at $0 \in U \subset \mathbb{R}^n$, we prove some general vanishing theorems on the poles of the local zeta function $Z_f(\varphi)(\lambda)$. First, let

$$f(x) = \sum_{\alpha \in \mathbb{Z}_+^n} a_\alpha x^\alpha \qquad (a_\alpha \in \mathbb{R}) \tag{3.2}$$

be the Taylor expansion of f at $0 \in \mathbb{R}^n$.

Definition 3.1. *(i) Let $\Gamma_+(f) \subset \mathbb{R}^n_+$ be the convex hull of $\bigcup_{\alpha: a_\alpha \neq 0}(\alpha + \mathbb{R}^n_+)$ in $\mathbb{R}^n_+ = \mathbb{R}^n_{\geq 0}$. We call $\Gamma_+(f)$ the Newton polygon of f.*
(ii) For each compact face $\gamma \prec \Gamma_+(f)$ of $\Gamma_+(f)$ we set

$$f_\gamma(x) = \sum_{\alpha \in \gamma \cap \mathbb{Z}^n_+} a_\alpha x^\alpha \in \mathbb{R}[x_1, x_2, \dots, x_n]. \tag{3.3}$$

We call $f_\gamma(x)$ the γ-part of f.

From now on, we assume that f satisfies the following condition.

Definition 3.2. *We say that f is non-degenerate if for any compact face $\gamma \prec \Gamma_+(f)$ of $\Gamma_+(f)$ we have:*

$$\left(\frac{\partial f_\gamma}{\partial x_1}(x), \frac{\partial f_\gamma}{\partial x_2}(x), \cdots\cdots, \frac{\partial f_\gamma}{\partial x_n}(x) \right) \neq (0, 0, \cdots\cdots, 0) \tag{3.4}$$

at any point x of $\{x \in \mathbb{R}^n \mid x_1 x_2 \cdots x_n \neq 0, f_\gamma(x) = 0\} \subset (\mathbb{R} \setminus \{0\})^n$.

For the sake of simplicity, let us assume also that f is convenient: for any $1 \leq i \leq n$ we have $\Gamma_+(f) \cap \{\alpha \in \mathbb{R}^n_+ \mid \alpha_j = 0 \text{ for } j \neq i \text{ and } \alpha_i > 0\} \neq \emptyset$. Then by the results of Varchenko [18] (see also [1] and [9]) we can describe the candidate poles of $Z_f(\varphi)$ in terms of $\Gamma_+(f)$ as follows. First Let Σ_0 be the dual fan of $\Gamma_+(f)$ and Σ a subdivision of Σ_0 such that the real toric variety X_Σ associated to it is smooth. Since $\Sigma = \{\sigma\}$ is a family of rational polyhedral convex cones in $\mathbb{R}^n_+$ such that $\mathbb{R}^n_+ = \cup_{\sigma \in \Sigma} \sigma$ there exists a natural proper morphism $\pi : X_\Sigma \longrightarrow \mathbb{R}^n$ of real analytic manifolds. By using the convenience of f, we can construct a smooth fan Σ such that π induces a diffeomorphism $X_\Sigma \setminus \pi^{-1}(0) \xrightarrow{\sim} \mathbb{R}^n \setminus \{0\}$. Moreover by the non-degeneracy of f, the pull-back $f \circ \pi : X_\Sigma \longrightarrow \mathbb{R}$ of f to X_Σ defines a hypersurface $\{f \circ \pi = 0\}$ in X_Σ with only normal crossing singularities. In order to describe $\pi : X_\Sigma \longrightarrow \mathbb{R}^n$ and $f \circ \pi$ we shall prepare some notations. By the smoothness of X_Σ, on any cone $\sigma \in \Sigma$ such that $\dim\sigma = k$ there exist exactly k edges (i.e. 1-dimensional faces). Let $a^1(\sigma), a^2(\sigma), \dots, a^k(\sigma) \in \partial\sigma \cap (\mathbb{Z}^n \setminus \{0\})$ be the non-zero primitive vectors on these edges of σ. We call $\{a^1(\sigma), a^2(\sigma), \dots, a^k(\sigma)\}$ the 1-skelton of σ. For each n-dimensional cone $\sigma \in \Sigma$ we fix the ordering of its 1-skelton $\{a^1(\sigma), a^2(\sigma), \dots, a^n(\sigma)\}$ so that the determinant of the invertible matrix

$$A(\sigma) = \{a^i(\sigma)_j\}_{i,j=1}^n \in GL_n(\mathbb{Z}) \tag{3.5}$$

is 1. For a cone $\sigma \in \Sigma$ and $1 \leq i \leq \dim\sigma$ we set

$$l(a^i(\sigma)) = \min_{\alpha \in \Gamma_+(f)} \langle a^i(\sigma), \alpha \rangle \in \mathbb{Z}_+ \tag{3.6}$$

and

$$|a^i(\sigma)| = \sum_{j=1}^n a^i(\sigma)_j \in \mathbb{Z}_{>0}. \tag{3.7}$$

Now let σ be an n-dimensional cone in Σ and $\mathbb{R}^n(\sigma) \simeq \mathbb{R}^n_y$ the affine open subset of X_Σ associated to σ. Then the restriction $\pi(\sigma) : \mathbb{R}^n(\sigma) \longrightarrow \mathbb{R}^n$ of $\pi : X_\Sigma \longrightarrow \mathbb{R}^n$ to $\mathbb{R}^n(\sigma) \simeq \mathbb{R}^n_y$ and its Jacobian $J(\pi(\sigma)) : \mathbb{R}^n(\sigma) \longrightarrow \mathbb{R}$ are explicitly given by

$$\pi(\sigma)(y) = \left(\prod_{i=1}^n y_i^{a^i(\sigma)_1}, \prod_{i=1}^n y_i^{a^i(\sigma)_2}, \cdots, \prod_{i=1}^n y_i^{a^i(\sigma)_n} \right), \tag{3.8}$$

$$J(\pi(\sigma))(y) = y_1^{|a^1(\sigma)|-1} y_2^{|a^2(\sigma)|-1} \cdots y_n^{|a^n(\sigma)|-1}. \tag{3.9}$$

Moreover on $\mathbb{R}^n(\sigma) \simeq \mathbb{R}^n_y$ we have

$$(f \circ \pi(\sigma))(y) = f_\sigma(y) \times \prod_{i=1}^{n} y_i^{l(a^i(\sigma))}, \tag{3.10}$$

where f_σ is a real analytic function defined on $\pi(\sigma)^{-1}(U) \subset \mathbb{R}^n(\sigma)$. By the non-degeneracy of f the hypersurface $\{f_\sigma = 0\}$ intersect all coordinate subspaces of $\mathbb{R}^n(\sigma)$ transversally. In particular, we have $f_\sigma(0) \neq 0$. For $0 \leq k \leq n$ let $\Sigma^{(k)} \subset \Sigma$ be the subset of Σ consisting of k-dimensional cones.

Definition 3.3. *(see [1], [9] and [18] etc.) Let $P \subset \mathbb{Q}_{<0}$ be the union of the following subsets of $\mathbb{Q}_{<0}$:*

$$\{-1, -2, -3, \cdots\cdots\}, \tag{3.11}$$

$$\left\{-\frac{|a^1(\sigma)|}{l(a^1(\sigma))}, -\frac{|a^1(\sigma)|+1}{l(a^1(\sigma))}, \cdots\cdots\right\} \qquad (\sigma \in \Sigma^{(1)} \text{ such that } l(a^1(\sigma)) > 0). \tag{3.12}$$

By the results of [18], the poles of the local zeta function $Z_f(\varphi)$ are contained in the set P. An element of P is called a candidate pole of $Z_f(\varphi)$. We order the candidate poles of $Z_f(\varphi)$ as

$$P = \{-\lambda_1 > -\lambda_2 > -\lambda_3 > \cdots\cdots\} \qquad (\lambda_j \in \mathbb{Q}_{>0}). \tag{3.13}$$

Hereafter we fix a candidate pole $-\lambda_j \in P$ of $Z_f(\varphi)$.

Definition 3.4. *(i) Let $\sigma \in \Sigma$. For $1 \leq i \leq \dim\sigma$ such that $l(a^i(\sigma)) > 0$ ($\Longleftrightarrow a^i(\sigma) \in \mathbb{R}^n_{>0}$) we set*

$$K^i(\sigma) = \left\{\frac{|a^i(\sigma)|}{l(a^i(\sigma))}, \frac{|a^i(\sigma)|+1}{l(a^i(\sigma))}, \cdots\cdots\right\} \subset \mathbb{Q}_{>0}. \tag{3.14}$$

(ii) For the candidate pole $-\lambda_j \in P$ of $Z_f(\varphi)$ and $0 \leq k \leq n$ we define a subset $\Sigma_j^{(k)}$ of $\Sigma^{(k)}$ by

$$\Sigma_j^{(k)} = \{\sigma \in \Sigma^{(k)} \mid l(a^i(\sigma)) > 0 \text{ and } \lambda_j \in K^i(\sigma) \text{ for } 1 \leq i \leq k\}. \tag{3.15}$$

(iii) For $\sigma \in \Sigma_j^{(k)}$ and $1 \leq i \leq k$ we define a non-negative integer $\nu(\sigma)_i \in \mathbb{Z}_+$ by

$$\lambda_j = \frac{|a^i(\sigma)| + \nu(\sigma)_i}{l(a^i(\sigma))}. \tag{3.16}$$

After [1] and [18] the following result is well-known to the specialists.

Theorem 3.5. *(i) Assume that $\lambda_j \notin \mathbb{Z}$. Then the order of the pole of $Z(\varphi)$ at $\lambda = -\lambda_j$ is less than or equal to*

$$k_j := \max\{0 \leq k \leq n \mid \Sigma_j^{(k)} \neq \emptyset\} \in \mathbb{Z}_+. \tag{3.17}$$

(ii) Assume that $\lambda_j \in \mathbb{Z}$. Then the order of the pole of $Z(\varphi)$ at $\lambda = -\lambda_j$ is less than or equal to

$$k_j := 1 + \max\{0 \leq k \leq n \mid \Sigma_j^{(k)} \neq \emptyset\} \in \mathbb{Z}_+. \tag{3.18}$$

Now let

$$\frac{a_{j,k_j}(\varphi)}{(\lambda+\lambda_j)^{k_j}}+\cdots\cdots+\frac{a_{j,2}(\varphi)}{(\lambda+\lambda_j)^2}+\frac{a_{j,1}(\varphi)}{(\lambda+\lambda_j)}+\cdots\cdots \qquad (a_{j,k}(\varphi)\in\mathbb{R}) \tag{3.19}$$

be the Laurent expansion of $Z_f(\varphi)$ at $\lambda=-\lambda_j$. Then we obtain the following result which generalizes that of Jacobs in [8].

Theorem 3.6. *(i) Assume that λ_j is not an odd integer and let $1\leq k\leq k_j$. Assume moreover that for any $\sigma\in\Sigma_j^{(k)}$ there exists $1\leq i\leq k$ such that $\nu(\sigma)_i$ is odd. Then we have $a_{j,k}(\varphi)=\cdots=a_{j,k_j}(\varphi)=0$.*
(ii) Assume that λ_j is an odd integer and let $2\leq k\leq k_j$. Assume moreover that for any $\sigma\in\Sigma_j^{(k-1)}$ there exists $1\leq i\leq k-1$ such that $\nu(\sigma)_i$ is odd. Then we have $a_{j,k}(\varphi)=\cdots=a_{j,k_j}(\varphi)=0$.

Proof. (i) Since suppφ is compact and $\pi: X_\Sigma\longrightarrow\mathbb{R}^n$ is proper, there exists finite C^∞-functions φ_q $(1\leq q\leq N)$ on X_Σ such that $\sum_{q=1}^N\varphi_q\equiv 1$ on $\mathrm{supp}(\varphi\circ\pi)$. We may assume that for any $1\leq q\leq N$ there exists an n-dimensional cone $\sigma_q\in\Sigma^{(n)}$ such that $\mathrm{supp}\varphi_q\subset\subset\mathbb{R}^n(\sigma_q)$. Then we have

$$\begin{aligned} &Z_f(\varphi)(\lambda)=\\ =\;&\sum_{q=1}^N\int_{\mathbb{R}^n(\sigma_q)}\prod_{i=1}^n|y_i|^{l(a^i(\sigma_q))\lambda+|a^i(\sigma_q)|-1}\times|f_{\sigma_q}|^\lambda(y)\times(\varphi\circ\pi(\sigma_q))(y)\varphi_q(y)dy. \end{aligned} \tag{3.20}$$

We divide the proof of (i) into the following two case.

(I) First assume that λ_j is not an integer. Then by (the proof of) Proposition 2.2, $a_{j,k}(\varphi)$ can be written as a

$$a_{j,k}(\varphi)=\sum_{q=1}^N\sum_{l\geq k}\sum_{\sigma\in\Sigma_{j,q}^{(l)}}J_q(\sigma), \tag{3.21}$$

where for $1\leq q\leq N$ and $0\leq l\leq n$ we set

$$\Sigma_{j,q}^{(l)}=\{\sigma\in\Sigma_j^{(l)}\mid\sigma\prec\sigma_q\}. \tag{3.22}$$

Moreover for l such that $k\leq l\leq n$ and $\sigma\in\Sigma_{j,q}^{(l)}$ the number $J_q(\sigma)$ is expressed as follows.

$$J_q(\sigma)=\frac{1}{(l-k)!}\times\frac{d^{l-k}}{d\lambda^{l-k}}\rho_{q,\sigma}(\lambda)|_{\lambda=-\lambda_j}. \tag{3.23}$$

Let us explain the function $\rho_{q,\sigma}(\lambda)$ which is holomorphic at $\lambda=-\lambda_j$. For the sake of simplicity, we assume that $\{a^1(\sigma_q),a^2(\sigma_q),\dots,a^l(\sigma_q)\}$ is the 1-skelton of $\sigma\prec\sigma_q$.

(a) (The case where $\mathrm{supp}\varphi_q\cap\{y\in\mathbb{R}^n(\sigma_q)\mid f_{\sigma_q}(y)=y_1=y_2=\cdots=y_l=0\}=\emptyset$) We set

$$\begin{aligned} \rho_{q,\sigma}(\lambda)\;&=\prod_{i=1}^l\frac{1+(-1)^{\nu(\sigma_q)_i}}{l(a^i(\sigma_q))\nu(\sigma_q)_i!}\int_{\{y_1=\cdots=y_l=0\}}\prod_{i=l+1}^n g_i(\lambda,y_i)\\ \times\;&\frac{\partial^{\nu(\sigma_q)_1+\cdots+\nu(\sigma_q)_l}}{\partial y_1^{\nu(\sigma_q)_1}\cdots\partial y_l^{\nu(\sigma_q)_l}}\left\{|f_{\sigma_q}|^\lambda(\varphi\circ\pi(\sigma_q))\varphi_q\right\}_{y_1=\cdots=y_l=0}dy_{l+1}\cdots dy_n, \end{aligned} \tag{3.24}$$

where $g_{l+1}(\lambda,\cdot),\ldots,g_n(\lambda,\cdot)$ are 1-dimensional integrable functions with holomorphic parameter λ at $\lambda=-\lambda_j\in P$.

(b) (The case where $\mathrm{supp}\varphi_q\cap\{y\in\mathbb{R}^n(\sigma_q)\mid f_{\sigma_q}(y)=y_1=y_2=\cdots=y_l=0\}\neq\emptyset$) For $1\leq i\leq n$ set $H_i=\{y\in\mathbb{R}^n(\sigma_q)\mid y_i=0\}$. Then we may assume that $\{1\leq i\leq n\mid \mathrm{supp}\varphi_q\cap H_i\neq\emptyset\}=\{1,2,\ldots,r\}$ for some $r\geq l$. In this case, by a real analytic local coordinate change $\Phi:(y_1,\ldots,y_n)\longmapsto(z_1,\ldots,z_n)$ such that $z_i=y_i$ $(1\leq i\leq r)$ which sends the hypersurface $\{f_{\sigma_q}=0\}$ to $\{z_n=0\}$, the function $\rho_{q,\sigma}(\lambda)$ is expressed as

$$\begin{aligned}\rho_{q,\sigma}(\lambda) &= \left(\prod_{i=1}^{l}\frac{1+(-1)^{\nu(\sigma_q)_i}}{l(a^i(\sigma_q))\nu(\sigma_q)_i!}\right)\int_{\{z_1=\cdots z_l=0\}}\left(\prod_{i=l+1}^{r}g_i(\lambda,z_i)\right)g_n(\lambda,z_n)\\ \frac{\partial^{\nu(\sigma_q)_1+\cdots+\nu(\sigma_q)_l}}{\partial z_1^{\nu(\sigma_q)_1}\cdots\partial z_l^{\nu(\sigma_q)_l}} &\left\{F(\lambda,z)(\varphi\circ\pi(\sigma_q)\circ\Phi^{-1})(\varphi_q\circ\Phi^{-1})\left|\frac{\partial(y_1,\ldots,y_n)}{\partial(z_1,\ldots,z_n)}\right|\right\}_{z_1=\cdots=z_l=0}\\ &dz_{l+1}\cdots dz_n, \end{aligned}\tag{3.25}$$

where

$$F(\lambda,z)=\left(\prod_{i=r+1}^{n}|y_i|^{l(a^i(\sigma_q))\lambda+|a^i(\sigma_q)|-1}\right)\circ\Phi^{-1}\tag{3.26}$$

is a real analytic function on a neighborhood of $\Phi(\mathrm{supp}\varphi_q)$ and $g_{l+1}(\lambda,\cdot)$, $\ldots$, $g_r(\lambda,\cdot)$ and $g_n(\lambda,\cdot)$ are 1-dimensional integrable functions with holomorphic parameter λ at $\lambda=-\lambda_j\in P$.

Now by our assumption, for any $\sigma\in\Sigma_{j,q}^{(l)}$ $(l\geq k)$ there exists $1\leq i\leq l$ such that $\nu(\sigma_q)_i$ is odd. Therefore we obtain $a_{j,k}(\varphi)=0$ in this case. In the same way, we can prove also that $a_{j,k+1}(\varphi)=\cdots a_{j,k_j}(\varphi)=0$.

(II) Next assume that λ_j is an integer and set $m:=\lambda_j$. Then by (the proof of) Proposition 2.2, $a_{j,k}(\varphi)$ can be written as

$$\sum_{q=1}^{N}\left\{\sum_{l\geq k}\sum_{\sigma\in\Sigma_{j,q}^{(l)}}J_q(\sigma)+\sum_{l\geq k-1}\sum_{\sigma\in\Sigma_{j,q}^{(l)}}\tilde{J}_q(\sigma)\right\},\tag{3.27}$$

where for l such that $k-1\leq l\leq n$ and $\sigma\in\Sigma_{j,q}^{(l)}$ the number $\tilde{J}_q(\sigma)$ is expressed as follows.

$$\tilde{J}_q(\sigma)=\frac{1}{(l+1-k)!}\times\frac{d^{l+1-k}}{d\lambda^{l+1-k}}\tau_{q,\sigma}(\lambda)|_{\lambda=-\lambda_j}.\tag{3.28}$$

Let us explain the function $\tau_{q,\sigma}(\lambda)$ which is holomorphic at $\lambda=-\lambda_j$. For simplicity, we assume that $\{a^1(\sigma_q),a^2(\sigma_q),\ldots,a^l(\sigma_q)\}$ is the 1-skelton of $\sigma\prec\sigma_q$. If $\mathrm{supp}\varphi_q\cap\{y\in\mathbb{R}^n(\sigma_q)\mid f_{\sigma_q}(y)=y_1=y_2=\cdots=y_l=0\}=\emptyset$ we set $\tau_{q,\sigma}(\lambda)\equiv 0$. If $\mathrm{supp}\varphi_q\cap\{y\in\mathbb{R}^n(\sigma_q)\mid f_{\sigma_q}(y)=y_1=y_2=\cdots=y_l=0\}\neq\emptyset$, assuming as in (b) that $\{1\leq i\leq n\mid\mathrm{supp}\varphi_q\cap H_i\neq\emptyset\}=\{1,2,\ldots,r\}$ for some $r\geq l$ and using the local coordinate change

Φ used in (b), the function $\tau_{q,\sigma}(\lambda)$ is expressed as

$$\begin{aligned}\tau_{q,\sigma}(\lambda) &= \left(\prod_{i=1}^{l} \frac{1+(-1)^{\nu(\sigma_q)_i}}{l(a^i(\sigma_q))\nu(\sigma_q)_i!}\right) \frac{1+(-1)^{m-1}}{(m-1)!} \int_{\{z_1=\cdots=z_l=z_n=0\}} \left(\prod_{i=l+1}^{r} g_i(\lambda, z_i)\right) \\ \frac{\partial^{\nu(\sigma_q)_1+\cdots+\nu(\sigma_q)_l+m-1}}{\partial z_1^{\nu(\sigma_q)_1}\cdots\partial z_l^{\nu(\sigma_q)_l}\partial z_n^{m-1}} & \left\{F(\lambda,z)(\varphi\circ\pi(\sigma_q)\circ\Phi^{-1})(\varphi_q\circ\Phi^{-1})|\frac{\partial(y_1,\ldots,y_n)}{\partial(z_1,\ldots,z_n)}|\right\}_{z_1=\cdots=z_l=z_n=0} \\ & dz_{l+1}\cdots dz_{n-1}, \end{aligned} \tag{3.29}$$

where $F(\lambda,z)$ and $g_{l+1}(\lambda,\cdot),\ldots,g_r(\lambda,\cdot)$ are as in (b). Then, as in (I), by our assumption we obatin $J_q(\sigma)=0$ for any $\sigma\in\Sigma_{j,q}^{(l)}$ $(l\geq k)$. Moreover, since by our assumption in (i) the integer $m=\lambda_j$ must be even, we obtain $\tilde{J}_q(\sigma)=0$ for any $\sigma\in\Sigma_{j,q}^{(l)}$ $(l\geq k-1)$. Therefore we get $a_{j,k}(\varphi)=0$ in this case, too. In the same way, we can prove also that $a_{j,k+1}(\varphi)=\cdots a_{j,k_j}(\varphi)=0$. This completes the proof of (i). The remaining assertion (ii) can be shown similarly. □

By this theorem we see that many candidate poles of $Z_f(\varphi)$ are fake, i.e. not actual poles. We can also find a nice condition on the test function $\varphi\in C_0^\infty(\mathbb{R}^n)$ under which we have the vanishing $a_{j,k}(\varphi)=\cdots=a_{j,k_j}(\varphi)=0$. For this purpose, we introduce the following subset $\Delta_{j,k}$ of $\mathbb{R}_+^n$.

Definition 3.7. *Let $1\leq k\leq k_j$.*
(i) For $\sigma\in\Sigma_j^{(k)}$ we define a compact convex subset $\Delta_{j,k}^\sigma$ of $\mathbb{R}_+^n$ by

$$\Delta_{j,k}^\sigma=\{\alpha\in\mathbb{R}_+^n \mid \langle a^i(\sigma),\alpha\rangle\leq\nu(\sigma)_i \quad \textit{for any} \quad 1\leq i\leq k\}. \tag{3.30}$$

(ii) We define a compact subset $\Delta_{j,k}$ of $\mathbb{R}_+^n$ by

$$\Delta_{j,k}=\bigcup_{\sigma\in\Sigma_j^{(k)}}\Delta_{j,k}^\sigma. \tag{3.31}$$

Note that $\Delta_{j,k}$ is not necessarily a convex subset of $\mathbb{R}_+^n$. It follows also from the definition that we have $\Delta_{j,k}\supset\Delta_{j,k'}$ for $1\leq k\leq k'\leq k_j$. In order to state our another vanishing theorem, let

$$\varphi(x)=\sum_{\alpha\in\mathbb{Z}_+^n}c_\alpha x^\alpha \qquad (c_\alpha\in\mathbb{R}) \tag{3.32}$$

be the Taylor expansion of the test function φ at $0\in U\subset\mathbb{R}^n$.

Theorem 3.8. *(i) Let $1\leq k\leq k_j$. Assume that λ_j is not an odd integer and $\{\alpha\in\mathbb{Z}_+^n \mid c_\alpha\neq0\}\cap\Delta_{j,k}=\emptyset$. Then we have $a_{j,k}(\varphi)=\cdots=a_{j,k_j}(\varphi)=0$.*
(ii) Let $2\leq k\leq k_j$. Assume that λ_j is an odd integer and $\{\alpha\in\mathbb{Z}_+^n \mid c_\alpha\neq0\}\cap\Delta_{j,k-1}=\emptyset$. Then we have $a_{j,k}(\varphi)=\cdots=a_{j,k_j}(\varphi)=0$.

Proof. We use the notations in the proof of Theorem 3.6.
(i) Since the proof for the case where λ_j is an integer is similar, we prove the assertion only in the case where λ_j is not an integer. In this case, we have

$$a_{j,k}(\varphi)=\sum_{q=1}^{N}\sum_{l\geq k}\sum_{\sigma\in\Sigma_{j,q}^{(l)}}\frac{1}{(l-k)!}\frac{d^{l-k}}{d\lambda^{l-k}}\rho_{q,\sigma}(\lambda)|_{\lambda=-\lambda_j}, \tag{3.33}$$

where the function $\rho_{q,\sigma}(\lambda)$ is holomorphic at $\lambda = -\lambda_j$ (for its expression, see the proof of Theorem 3.6). For $\alpha \in \mathbb{Z}_+^n$ let $\psi_\alpha \in C_0^\infty(U)$ be a test function on U such that $\psi_\alpha \equiv x^\alpha$ in a neighborhood of $0 \in U \subset \mathbb{R}^n$. For $1 \leq q \leq N$ and $\sigma \in \Sigma_{j,q}^{(l)}$ $(l \geq k)$, assume for simplicity that $\{a^1(\sigma_q), a^2(\sigma_q), \ldots, a^l(\sigma_q)\}$ is the 1-skelton of $\sigma \prec \sigma_q$. Then in an open neighborhood of $\{y \in \mathbb{R}^n(\sigma_q) \mid y_1 = \cdots = y_l = 0\} \subset \mathbb{R}^n(\sigma_q) \subset X_\Sigma$ we have

$$(\psi_\alpha \circ \pi(\sigma_q))(y) \equiv y_1^{\langle a^1(\sigma_q),\alpha\rangle} \cdots y_n^{\langle a^n(\sigma_q),\alpha\rangle}. \tag{3.34}$$

Moreover, by the definition of $\Delta_{j,k}$, if $\alpha \notin \Delta_{j,k}$ then we have $\alpha \notin \Delta_{j,l}$ and there exists $1 \leq i \leq l$ such that $\langle a^i(\sigma_q), \alpha\rangle > \nu(\sigma_q)_i$. This implies the vanishing of the function

$$\frac{\partial^{\nu(\sigma_q)_1+\cdots+\nu(\sigma_q)_l}}{\partial y_1^{\nu(\sigma_q)_1}\cdots \partial y_l^{\nu(\sigma_q)_l}} \left\{|f_{\sigma_q}|^\lambda (\psi_\alpha \circ \pi(\sigma_q))\varphi_q\right\}_{y_1=\cdots=y_l=0} \equiv 0. \tag{3.35}$$

By applying this vanishing result to the expression of $\rho_{q,\sigma}(\lambda)$ (in the proof of Theorem 3.6), we see that if $\{\alpha \in \mathbb{Z}_+^n \mid c_\alpha \neq 0\} \cap \Delta_{j,k} = \emptyset$ the vanishing $a_{j,k}(\varphi) = 0$ holds. In the same way, we can prove also that $a_{j,k+1}(\varphi) = \cdots a_{j,k_j}(\varphi) = 0$. This completes the proof of (i). The assertion (ii) can be shown similarly. □

Now let us consider the following two local zeta functions.

$$Z_f^\pm(\varphi)(\lambda) = \int_{\mathbb{R}^n} f_\pm^\lambda(x)\varphi(x)dx. \tag{3.36}$$

Note that we have $Z_f(\varphi) = Z_f^+(\varphi) + Z_f^-(\varphi)$. Then the poles of these lcoal zeta functions $Z_f^\pm(\varphi)$ are also contained in P and their Laurent expansions at a candidate pole $\lambda = -\lambda_j \in P$ have the following form:

$$\frac{a_{j,k_j}^\pm(\varphi)}{(\lambda+\lambda_j)^{k_j}} + \cdots\cdots + \frac{a_{j,2}^\pm(\varphi)}{(\lambda+\lambda_j)^2} + \frac{a_{j,1}^\pm(\varphi)}{(\lambda+\lambda_j)} + \cdots\cdots \qquad (a_{j,k}^\pm(\varphi) \in \mathbb{R}) \tag{3.37}$$

(see for example [1], [9] etc.). By the proof of Theorem 3.8 we obtain a vanishing theorem also for the coefficients $a_{j,k}^\pm(\varphi)$ of the poles of $Z_f^\pm(\varphi)$.

Theorem 3.9. *(i) Let $1 \leq k \leq k_j$. Assume that λ_j is not an integer and $\{\alpha \in \mathbb{Z}_+^n \mid c_\alpha \neq 0\} \cap \Delta_{j,k} = \emptyset$. Then we have $a_{j,k}^\pm(\varphi) = \cdots = a_{j,k_j}^\pm(\varphi) = 0$.*
(ii) Let $2 \leq k \leq k_j$. Assume that λ_j is an integer and $\{\alpha \in \mathbb{Z}_+^n \mid c_\alpha \neq 0\} \cap \Delta_{j,k-1} = \emptyset$. Then we have $a_{j,k}^\pm(\varphi) = \cdots = a_{j,k_j}^\pm(\varphi) = 0$.

4 Explicit formulas for the poles of local zeta functions

In this section we give some explicit formulas for the coefficients $a_{j,n}(\varphi)$, $a_{j,n}^\pm(\varphi)$ of the deepest poles $\lambda = -\lambda_j \in P$ of the local zeta functions $Z_f(\varphi)$, $Z_f^\pm(\varphi)$ introduced in Section 3. We inherit the situation and the notations in Section 3. Let $-\lambda_j \in P$ be a candidate pole of $Z_f(\varphi)$.

Definition 4.1. *For $\sigma \in \Sigma_j^{(n)}$ and $\alpha \in \mathbb{Z}_+^n$ we define an integer $\mu(\sigma,\alpha)_i$ by*

$$\mu(\sigma,\alpha)_i = \nu(\sigma)_i - \langle a^i(\sigma), \alpha\rangle \in \mathbb{Z}. \tag{4.1}$$

Theorem 4.2. *Assume that λ_j is not an odd integer and $k_j = n$. Then the coefficient $a_{j,n}(\varphi)$ of the deepest possible pole $\lambda = -\lambda_j \in P$ of $Z_f(\varphi)$ is given by*

$$\begin{aligned} & a_{j,n}(\varphi) \\ = & \sum_{\alpha\in\Delta_{j,n}} \left\{ \sum_{\sigma\in\Sigma_j^{(n)}} \left(\prod_{i=1}^n \frac{1+(-1)^{\nu(\sigma)_i}}{l(a^i(\sigma)) \times \mu(\sigma,\alpha)_i!} \right) \frac{\partial^{\mu(\sigma,\alpha)_1+\cdots+\mu(\sigma,\alpha)_n}}{\partial y_1^{\mu(\sigma,\alpha)_1}\cdots\partial y_n^{\mu(\sigma,\alpha)_n}} |f_\sigma|^{-\lambda_j}(0) \right\} \\ \times & \frac{\partial_x^\alpha \varphi(0)}{\alpha!}, \end{aligned} \tag{4.2}$$

where for $\mu < 0$ we set $\frac{\partial^\mu}{\partial y_i^\mu}(\cdot) = 0$.

Proof. Since λ_j is not an odd integer, by the proof of Theorem 3.6 we have

$$\begin{aligned} a_{j,n}(\varphi) = & \sum_{\sigma\in\Sigma_j^{(n)}} \left(\prod_{i=1}^n \frac{1+(-1)^{\nu(\sigma)_i}}{l(a^i(\sigma)) \times \nu(\sigma)_i!} \right) \\ & \times \frac{\partial^{\nu(\sigma)_1+\cdots+\nu(\sigma)_n}}{\partial y_1^{\nu(\sigma)_1}\cdots\partial y_n^{\nu(\sigma)_n}} \left\{ |f_\sigma|^{-\lambda_j}(\varphi\circ\pi(\sigma)) \right\}_{y=0}. \end{aligned} \tag{4.3}$$

Now let

$$\varphi(x) = \sum_{\alpha\in\mathbb{Z}_+^n} \frac{\partial_x^\alpha \varphi(0)}{\alpha!} x^\alpha \tag{4.4}$$

be the Taylor expansion of φ at $0 \in \mathbb{R}^n$. Since $a_{j,n}(x^\alpha) = 0$ for $\alpha \notin \Delta_{j,n}$ by Theorem 3.8 and we have

$$(x^\alpha \circ \pi(\sigma))(y) \equiv y_1^{\langle a^1(\sigma),\alpha\rangle} \cdots y_n^{\langle a^n(\sigma),\alpha\rangle} \tag{4.5}$$

in a neighborhood of $0 \in \mathbb{R}^n(\sigma)$ for any $\sigma \in \Sigma_j^{(n)}$, we obtain

$$\begin{aligned} a_{j,n}(\varphi) = & \sum_{\alpha\in\Delta_{j,n}} \\ & \left\{ \sum_{\sigma\in\Sigma_j^{(n)}} \left(\prod_{i=1}^n \frac{1+(-1)^{\nu(\sigma)_i}}{l(a^i(\sigma)) \times \nu(\sigma)_i!} \right) \frac{\partial^{\nu(\sigma)_1+\cdots+\nu(\sigma)_n}}{\partial y_1^{\nu(\sigma)_1}\cdots\partial y_n^{\nu(\sigma)_n}} \left(|f_\sigma|^{-\lambda_j} y_1^{\langle a^1(\sigma),\alpha\rangle} \cdots y_n^{\langle a^n(\sigma),\alpha\rangle} \right)_{y=0} \right\} \\ & \times \frac{\partial_x^\alpha \varphi(0)}{\alpha!}. \end{aligned} \tag{4.6}$$

Then the result follows from the Leibniz rule. This completes the proof. □

In order to state similar results for $a_{j,n}^{\pm}(\varphi)$ we define two integers $c_\pm(\sigma)$ ($\sigma \in \Sigma_j^{(n)}$) as follows. First set $\{\pm1\}^n := \{\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \mid \varepsilon_i = \pm1\}$. For $\sigma \in \Sigma_j^{(n)}$ we define subset $Q_\pm(\sigma)$ of $\{\pm1\}^n$ by

$$Q_\pm(\sigma) = \{\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \mid \pm f_\sigma(0) \times \prod_{i=1}^n \varepsilon_i^{l(a^i(\sigma))} > 0\}. \tag{4.7}$$

Let us explain the meaning of $Q_\pm(\sigma) \subset \{\pm 1\}^n$. For each $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{\pm 1\}^n$ we define an open subset V_ε of $\mathbb{R}^n(\sigma) \simeq \mathbb{R}^n_y$ by

$$V_\varepsilon = \{(y_1, y_2, \dots, y_n) \in \mathbb{R}^n(\sigma) \mid \varepsilon_i y_i > 0 \text{ for any } 1 \le i \le n\}. \tag{4.8}$$

Then there exists a sufficiently small open neighborhood W of $0 \in \mathbb{R}^n(\sigma)$ such that $\pm(f \circ \pi(\sigma))|_{W \cap V_\varepsilon} > 0$ for any $\varepsilon \in Q_\pm(\sigma)$. Namely $Q_\pm(\sigma)$ is naturally identified with the set $\{V_\varepsilon\}_{\varepsilon \in Q_\pm(\sigma)}$ of open quadrants in $\mathbb{R}^n(\sigma) \simeq \mathbb{R}^n_y$ such that $\pm(f \circ \pi(\sigma))|_{W \cap V_\varepsilon} > 0$ in a neighborhood of $0 \in \mathbb{R}^n(\sigma)$.

Definition 4.3. *For $\sigma \in \Sigma_j^{(n)}$ we set*

$$c_\pm(\sigma) = \sum_{\varepsilon \in Q_\pm(\sigma)} \left(\prod_{i=1}^n \varepsilon_i^{\nu(\sigma)_i} \right) \in \mathbb{Z}. \tag{4.9}$$

Note that for any $\sigma \in \Sigma_j^{(n)}$ we have

$$c_+(\sigma) + c_-(\sigma) = \prod_{i=1}^n \{1 + (-1)^{\nu(\sigma)_i}\}. \tag{4.10}$$

Theorem 4.4. *Assume that λ_j is not an integer and $k_j = n$. Then the coefficient $a^\pm_{j,n}(\varphi)$ of the deepest possible pole $\lambda = -\lambda_j \in P$ of $Z_f^\pm(\varphi)$ is given by*

$$\begin{aligned} & a^\pm_{j,n}(\varphi) \\ = & \sum_{\alpha \in \Delta_{j,n}} \left\{ \sum_{\sigma \in \Sigma_j^{(n)}} c_\pm(\sigma) \left(\prod_{i=1}^n \frac{1}{l(a^i(\sigma)) \times \mu(\sigma,\alpha)_i!} \right) \frac{\partial^{\mu(\sigma,\alpha)_1 + \cdots + \mu(\sigma,\alpha)_n}}{\partial y_1^{\mu(\sigma,\alpha)_1} \cdots \partial y_n^{\mu(\sigma,\alpha)_n}} |f_\sigma|^{-\lambda_j}(0) \right\} \\ & \times \frac{\partial_x^\alpha \varphi(0)}{\alpha!}, \end{aligned} \tag{4.11}$$

where for $\mu < 0$ we set $\frac{\partial^\mu}{\partial y_i^\mu}(\cdot) = 0$.

5 Asymptotic expansions of oscillating integrals

In this section, combining our previous arguments with those of [1] and [18], we obtain some results on the asymptotic expansions of oscillating integrals. As before, let f be a real-valued real analytic function defined on an open neighborhood U of $0 \in \mathbb{R}^n$ and $\varphi \in C_0^\infty(U)$ a real-valued test function defined on U. We inherit the notations in Section 3 and 4. Then the oscillating integral $I_f(\varphi)(t)$ $(t \in \mathbb{R})$ associated to f and φ is defined by

$$I_f(\varphi)(t) = \int_{\mathbb{R}^n} e^{itf(x)} \varphi(x) dx. \tag{5.1}$$

Here we set $i = \sqrt{-1}$ for short. By the fundamental results of Varchenko [18] (see also [1] and [9] for the detail), as $t \longrightarrow +\infty$ the oscillating integral $I_f(\varphi)(t)$ has an asymptotic expansion of the form

$$I_f(\varphi)(t) \sim \sum_{j=1}^\infty \sum_{k=1}^{k_j} c_{j,k}(\varphi) t^{-\lambda_j} (\log t)^{k-1}, \tag{5.2}$$

where $c_{j,k}(\varphi)$ are some complex numbers. Despite the important contributions by many mathematicians (see for example [1], [2], [6] and [17] etc.), little is known about the coefficients $c_{j,k}(\varphi)$ of the asymptotic expansion. First of all, we shall give a general vanishing theorem for these coefficients $c_{j,k}(\varphi)$. Let us fix a candidate pole $-\lambda_j \in P$ of the local zeta function $Z_f(\varphi)$ and let

$$\varphi(x) = \sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha x^\alpha \qquad (c_\alpha \in \mathbb{R}) \tag{5.3}$$

be the Taylor expansion of the test function φ at $0 \in U \subset \mathbb{R}^n$.

Theorem 5.1. *(i) Let $1 \leq k \leq k_j$. Assume that λ_j is not an integer and $\{\alpha \in \mathbb{Z}_+^n \mid c_\alpha \neq 0\} \cap \Delta_{j,k} = \emptyset$. Then we have $c_{j,k}(\varphi) = \cdots = c_{j,k_j}(\varphi) = 0$.*
(ii) Let $2 \leq k \leq k_j$. Assume that λ_j is an integer and $\{\alpha \in \mathbb{Z}_+^n \mid c_\alpha \neq 0\} \cap \Delta_{j,k-1} = \emptyset$. Then we have $c_{j,k}(\varphi) = \cdots = c_{j,k_j}(\varphi) = 0$.

Proof. By using the results of [1] and [18], the result follows immediately from Theorem 3.9. □

Next we give an explicit formula for the coefficient $c_{j,n}(\varphi)$ of $t^{-\lambda_j}(\log t)^{n-1}$ in the asymptotic expansion (5.2). For this purpose, we define two real numbers $b_{j,n}^{\pm}(\varphi) \in \mathbb{R}$ by

$$\begin{aligned} & b_{j,n}^{\pm}(\varphi) \\ = \quad & \sum_{\alpha \in \Delta_{j,n}} \left\{ \sum_{\sigma \in \Sigma_j^{(n)}} c_{\pm}(\sigma) \left(\prod_{i=1}^{n} \frac{1}{l(a^i(\sigma))\mu(\sigma,\alpha)_i!} \right) \frac{\partial^{\mu(\sigma,\alpha)_1 + \cdots + \mu(\sigma,\alpha)_n}}{\partial y_1^{\mu(\sigma,\alpha)_1} \cdots \partial y_n^{\mu(\sigma,\alpha)_n}} |f_\sigma|^{-\lambda_j}(0) \right\} \\ & \times \frac{\partial_x^\alpha \varphi(0)}{\alpha!}, \end{aligned} \tag{5.4}$$

where for $\mu < 0$ we set $\frac{\partial^\mu}{\partial y_i^\mu}(\cdot) = 0$. Recall that if λ_j is not an integer we have $a_{j,n}^{\pm}(\varphi) = b_{j,n}^{\pm}(\varphi)$.

Theorem 5.2. *The coefficient $c_{j,n}(\varphi)$ of $t^{-\lambda_j}(\log t)^{n-1}$ in the asymptotic expansion* (5.2) *of $I_f(\varphi)$ is given by*

$$c_{j,n}(\varphi) = \frac{\Gamma(\lambda_j)}{(n-1)!} \left(e^{\frac{\pi i}{2}\lambda_j} b_{j,n}^{+}(\varphi) + e^{-\frac{\pi i}{2}\lambda_j} b_{j,n}^{-}(\varphi) \right). \tag{5.5}$$

Proof. We use the notations in the proof of Theorem 3.6. By the arguments in [1] and [9] etc. we do not have to consider the contributions from $\{f_{\sigma_q} = 0\}$ $(q = 1, 2, \ldots, N)$. Then the result follows from (the proof of) Theorem 4.2 and 4.4. □

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